

**ANALYSIS OF BOUNDARY-DOMAIN INTEGRAL  
EQUATIONS BASED ON A NEW PARAMETRIX FOR  
THE MIXED DIFFUSION BVP WITH VARIABLE  
COEFFICIENT IN AN INTERIOR LIPSCHITZ  
DOMAIN**

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ABSTRACT. A mixed boundary value problem for the partial differential equation of diffusion in an inhomogeneous medium in a Lipschitz domain is reduced to a system of direct segregated parametrix-based Boundary-Domain Integral Equations (BDIEs). We use a parametrix different from the one employed in the papers by Mikhailov (2002, 2006) and Chkadua, Mikhailov, Natroshvili (2009). We prove the equivalence between the original BVP and the corresponding BDIE system. The invertibility and Fredholm properties of the boundary-domain integral operators are also analysed.

**1. Introduction.** Boundary Domain Integral Equations (BDIEs) associated with variable-coefficient PDEs were studied in [2] for a scalar mixed elliptic BVP in bounded domains, [4, 21] for the corresponding problem in unbounded domains, [19] for the mixed problem for the incompressible Stokes system in bounded domains, [22, 23] for the mixed problem for the Compressible Stokes in bounded domains and [7] for a 2D mixed elliptic problem in bounded domains. Further results on the theory of BDIEs for BVPs with variable coefficient can be found on [20, 13, 14, 15, 9, 3, 7, 1]. Let us note that these types of BVPs model, for example, the heat transfer in inhomogeneous media or the motion of a laminar fluid with variable viscosity.

The BDIE systems can be solved numerically after discretising them e.g. by the collocation method, cf. [17, 25, 26], which leads to the linear algebraic systems with fully populated matrices. The method performance is essentially improved by implementing hierarchical matrix compression technique in conjunction with the adaptive cross ap-

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proximation procedure to [9] and iterative methods, cf. [27]. Another option is to discretise the *localised* version of BDIEs, based on the localised parametrics, which leads to systems of linear algebraic equations with sparse matrices [18, 28, 25, 26].

In order to deduce a BDIE system for a BVP with variable coefficients, usually a parametrix (Levi function) strongly related with the fundamental solution of the corresponding PDE with constant coefficients is employed. Using this relation, it is possible to establish further relations between the surface and volume potential type operators for the variable-coefficient case with their counterparts for the constant coefficient case, see, e.g. [2, Eq. (3.10)-(3.13)], [19, Eq. (34.10)-(34.16)].

For the scalar operator

$$(1) \quad Au(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x)}{\partial x_i} \right),$$

a parametrics

$$P^y(x, y) = P(x, y; a(y)) = \frac{-1}{4\pi a(y)|x - y|}$$

has been employed in [2, 3, 4], where  $x$  is the integration variable in the parametrix-based integral potentials. Note that the superscript in  $P^y(x, y)$  means that the parametrix is expressed in terms of the variable coefficient at point  $y$ .

There are many different ways of constructing parametrics and corresponding parametrix-based potentials and BDIEs, for the same variable-coefficient PDE, and performance of the BDIE-based numerical methods essentially depends on the chosen parametrix. To optimise the numerical method, it is beneficial to analyse the BDIEs based on different parametrics. It appeared, however, that not always the corresponding parametrix-based potentials and BDIEs can be easily analysed. The main motivation of this paper is to extend the collection of parametrics for which the analysis of the parametrix-based potentials and BDIEs is tractable. This will then allow to chose the tractable parametrics with more preferable properties, e.g., for numerical implementation. To this end, we employ *in this paper* the parametrix

$$P^x(x, y) = P(x, y; a(x)) = \frac{-1}{4\pi a(x)|x - y|}$$

for the same operator  $A$  defined by (1), where  $x$  is again the integration variable in the parametrix-based integral potentials.

Different families of parametrices lead to different relations with their counterparts for the constant coefficient case. For the parametrices considered in this paper these relations are rather simple, which makes it possible to obtain the mapping properties of the integral potentials in Sobolev spaces and prove the equivalence between the BDIE system and the BVP. After studying the Fredholm properties of the matrix operator which defines the systems, their invertibility is proved, which implies the uniqueness of solution of the BDIE system.

**2. Preliminaries and the BVP.** Let  $\Omega = \Omega^+$  be a bounded simply connected open Lipschitz domain,  $\Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$  the complementary (unbounded) domain. The Lipschitz boundary  $\partial\Omega$  is connected and closed. Furthermore,  $\partial\Omega := \overline{\partial\Omega_N} \cup \overline{\partial\Omega_D}$  where both  $\partial\Omega_N$  and  $\partial\Omega_D$  are non-empty, connected disjoint Lipschitz submanifolds of  $\partial\Omega$  with a Lipschitz interface between them.

Let us consider the partial differential equation

$$(2) \quad Au(x) := \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( a(x) \frac{\partial u(x)}{\partial x_i} \right) = f(x), \quad x \in \Omega,$$

where the variable smooth coefficient  $a(x) \in C^\infty(\overline{\Omega})$  is such that

$$(3) \quad 0 < a_{\min} \leq a(x) \leq a_{\max} < \infty, \quad \forall x \in \overline{\Omega},$$

$u(x)$  is an unknown function and  $f$  is a given function on  $\Omega$ . It is easy to see that if  $a \equiv 1$  then, the operator  $A$  becomes the Laplace operator,  $\Delta$ .

We will use the following function spaces in this paper (see, e.g., [11, 12] for more details). Let  $\mathcal{D}'(\Omega)$  be the Schwartz distribution space;  $H^s(\Omega)$  and  $H^s(\partial\Omega)$  with  $s \in \mathbb{R}$  be the Bessel potential spaces; the space  $\tilde{H}^s(\Omega)$  consisting of all the distributions of  $H^s(\mathbb{R}^3)$  whose support belongs to the closed set  $\overline{\Omega}$ . The corresponding spaces in  $\Omega^-$  are defined similarly. We will also need the following spaces on a Lipschitz submanifold  $S$  of  $\partial\Omega$ :

$$\begin{aligned} \tilde{H}^s(S) &:= \{g \in H^s(\partial\Omega) : \text{supp}(g) \subset \overline{S}\}, \\ H^s(S) &:= \{r_S g : g \in H^s(\partial\Omega)\}, \end{aligned}$$

where the notation  $r_S g$  is used for the restriction of the function  $g$  from  $\partial\Omega$  to  $S$ . We will also make use of the space

$$H^{1,0}(\Omega; A) := \{u \in H^1(\Omega) : Au \in L^2(\Omega)\},$$

see, e.g., [8, 5, 15], which is a Hilbert space with the norm defined by

$$\|u\|_{H^{1,0}(\Omega; A)}^2 := \|u\|_{H^1(\Omega)}^2 + \|Au\|_{L^2(\Omega)}^2.$$

**Traces and conormal derivatives.** For a scalar function  $w \in H^s(\Omega^\pm)$ ,  $1/2 < s$ , the traces  $\gamma^\pm w \in H^{s-\frac{1}{2}}(\partial\Omega)$  on the Lipschitz boundary  $\partial\Omega$  are well defined. Moreover, if  $1/2 < s < 3/2$ , the corresponding trace operators  $\gamma^\pm := \gamma_{\partial\Omega}^\pm : H^s(\Omega^\pm) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega)$  are continuous (see, e.g., [12, 15]).

For  $u \in H^s(\Omega)$ ,  $s > 3/2$ , we can define on  $\partial\Omega$  the conormal derivative operator,  $T^\pm$ , in the classical (trace) sense

$$T_x^\pm u := \sum_{i=1}^3 a(x) \gamma^\pm \left( \frac{\partial u}{\partial x_i} \right) n_i^\pm(x),$$

where  $n^+(x)$  is the exterior unit normal vector directed *outwards* the interior domain  $\Omega$  at a point  $x \in \partial\Omega$ . Similarly,  $n^-(x)$  is the unit normal vector directed *inwards* the interior domain  $\Omega$  at a point  $x \in \partial\Omega$ . Sometimes we will also use the notation  $T_x^\pm u$  or  $T_y^\pm u$  to emphasise which respect to which variable we are differentiating. Note that when the variable coefficient  $a \equiv 1$ , the operator  $T^\pm$  becomes the classical normal derivative  $\partial_n^\pm$ .

Moreover, for any function  $u \in H^{1,0}(\Omega; A)$ , the *canonical* conormal derivative  $T^\pm u \in H^{-\frac{1}{2}}(\Omega)$ , is well defined, cf. [5, 12, 15],

$$(4) \quad \langle T^\pm u, w \rangle_{\partial\Omega} := \pm \int_{\Omega^\pm} [(\gamma^{-1} w) Au + E(u, \gamma^{-1} w)] dx, \quad w \in H^{\frac{1}{2}}(\partial\Omega),$$

where  $\gamma^{-1} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\mathbb{R}^3)$  is a continuous right inverse to the trace operator whereas the function  $E$  is defined as

$$E(u, v)(x) := \sum_{i=1}^3 a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i},$$

and  $\langle \cdot, \cdot \rangle_{\partial\Omega}$  represents the  $L^2$ -based dual form on  $\partial\Omega$ .

We aim to derive boundary-domain integral equation systems for the following *mixed* boundary value problem. Given  $f \in L^2(\Omega)$ ,  $\phi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$  and  $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ , we seek a function  $u \in H^1(\Omega)$  such that

$$(5a) \quad Au = f, \quad \text{in } \Omega;$$

$$(5b) \quad r_{\partial\Omega_D}\gamma^+u = \phi_0, \quad \text{on } \partial\Omega_D;$$

$$(5c) \quad r_{\partial\Omega_N}T^+u = \psi_0, \quad \text{on } \partial\Omega_N;$$

where equation (5a) is understood in the weak sense, the Dirichlet condition (5b) is understood in the trace sense, the Neumann condition (5c) is understood in the functional sense (4),  $r_{\partial\Omega_D}$  and  $r_{\partial\Omega_N}$  are restrictions of the functions (or distributions) from  $\partial\Omega$  to  $\partial\Omega_D$  and  $\partial\Omega_N$ , respectively.

By Lemma 3.4 of [5] (cf. also Theorem 3.9 in [15] for a more general case), the first Green identity holds for any  $u \in H^{1,0}(\Omega; A)$  and  $v \in H^1(\Omega)$ ,

$$(6) \quad \langle T^\pm u, \gamma^\pm v \rangle_{\partial\Omega} := \pm \int_{\Omega} [vAu + E(u, v)] dx.$$

The following assertion is well known and can be proved, e.g., using the Lax-Milgram lemma as e.g. in [29, Theorem 4.11].

**Theorem 1.** *If the coefficient  $a$  satisfies condition (3), then the mixed problem (5) has one and only one solution in  $H^1(\Omega)$ .*

**3. Parametrices and remainders.** For the differential operator  $A$  presented in (1), we define a parametrix (Levi function)  $P(x, y)$  as a function of two (vector) variables  $x$  and  $y$  such that

$$(7) \quad A_x P(x, y) = \delta(x - y) + R(x, y),$$

where the notation  $A_x$  indicates differentiating with respect to  $x$ , while the function  $R(x, y)$  has at most weak singularity when  $x = y$ . For a given operator  $A$ , the parametrix is not unique. For example, the parametrix

$$P^y(x, y) = \frac{1}{a(y)} P_\Delta(x - y), \quad x, y \in \mathbb{R}^3,$$

was employed in [13, 2, 16], for the operator  $A$  defined in (1), where

$$P_{\Delta}(x-y) = \frac{-1}{4\pi|x-y|}$$

is the fundamental solution of the Laplace operator. The remainder corresponding to the parametrix  $P^y$  is

$$R^y(x, y) = \sum_{i=1}^3 \frac{1}{a(y)} \frac{\partial a(x)}{\partial x_i} \frac{\partial}{\partial x_i} P_{\Delta}(x-y), \quad x, y \in \mathbb{R}^3.$$

In this paper, for the same operator  $A$ , we will use another parametrix,

$$(8) \quad P(x, y) := P^x(x, y) = \frac{1}{a(x)} P_{\Delta}(x-y), \quad x, y \in \mathbb{R}^3,$$

which leads to the corresponding remainder

$$(9) \quad \begin{aligned} R(x, y) = R^x(x, y) &= - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( \frac{1}{a(x)} \frac{\partial a(x)}{\partial x_i} P_{\Delta}(x, y) \right) \\ &= - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \left( \frac{\partial \ln a(x)}{\partial x_i} P_{\Delta}(x, y) \right), \quad x, y \in \mathbb{R}^3. \end{aligned}$$

Note that if the variable coefficient  $a$  is smooth enough, then

$$R^x(x, y), R^y(x, y) \in \mathcal{O}(|x-y|^{-2}) \text{ as } x \rightarrow y,$$

i.e., the both remainders  $R_x$  and  $R_y$  are indeed weakly singular.

**4. Volume and surface potentials.** For the function  $g$  defined on a domain  $\Omega_+ \subset \mathbb{R}^n$ , e.g.,  $\rho \in \mathcal{D}(\bar{\Omega})$ , the volume parametrix-based Newton-type potential and the remainder potential are respectively defined, for  $y \in \mathbb{R}^3$ , as

$$\begin{aligned} \mathcal{P}\rho(y) &:= \langle P(\cdot, y), g \rangle_{\Omega} = \int_{\Omega} P(x, y) \rho(x) dx \\ \mathcal{R}\rho(y) &:= \langle R(\cdot, y), g \rangle_{\Omega} = \int_{\Omega} R(x, y) \rho(x) dx. \end{aligned}$$

From definitions (8), (9), the operators  $\mathcal{P}$  and  $\mathcal{R}$  can be expressed in terms the Newtonian potential associated with the Laplace operator,

$$(10) \quad \mathcal{P}\rho = \mathcal{P}_\Delta \left( \frac{\rho}{a} \right),$$

$$(11) \quad \mathcal{R}\rho = \nabla \cdot [\mathcal{P}_\Delta(\rho \nabla \ln a)] - \mathcal{P}_\Delta(\rho \Delta \ln a).$$

Relations (10) and (11) will be used also to determine the operators  $\mathcal{P}$  and  $\mathcal{R}$  also for more general spaces for  $\rho$  and to obtain, similar to [16, Theorem 3.2], the following mapping properties of the parametrix-based volume operators from the well-known (cf., e.g., [5]) properties of the Newtonian potential associated with the Laplace equation.

**Theorem 2.** *Let  $s \in \mathbb{R}$ . Then, the following operators are continuous,*

$$(12) \quad \mathcal{P} : \tilde{H}^s(\Omega) \rightarrow H^{s+2}(\Omega), \quad s \in \mathbb{R},$$

$$(13) \quad \mathcal{P} : H^s(\Omega) \rightarrow H^{s+2}(\Omega), \quad -\frac{1}{2} < s < \frac{1}{2},$$

$$(14) \quad \mathcal{P} : L_2(\Omega) \rightarrow H^{2,0}(\Omega; A),$$

$$(15) \quad \mathcal{R} : \tilde{H}^s(\Omega) \rightarrow H^{s+1}(\Omega), \quad s \in \mathbb{R},$$

$$(16) \quad \mathcal{R} : H^s(\Omega) \rightarrow H^{s+1}(\Omega), \quad -\frac{1}{2} < s < \frac{1}{2},$$

$$(17) \quad \mathcal{R} : H^1(\Omega) \rightarrow H^{1,0}(\Omega; A).$$

Moreover, for  $\frac{1}{2} < s < \frac{3}{2}$ , the following operators are compact,

$$\mathcal{R} : H^s(\Omega) \rightarrow H^s(\Omega),$$

$$r_{S_1} \gamma^+ \mathcal{R} : H^s(\Omega) \rightarrow H^{s-\frac{1}{2}}(S_1),$$

$$r_{S_1} T^+ \mathcal{R} : H^s(\Omega) \rightarrow H^{s-\frac{3}{2}}(S_1).$$

The parametrix-based single layer and double layer surface potentials are defined for  $y \in \mathbb{R}^3 : y \notin \partial\Omega$ , as

$$V\rho(y) := - \int_{\partial\Omega} P(x, y) \rho(x) dS(x),$$

$$W\rho(y) := - \int_{\partial\Omega} T_x^+ P(x, y) \rho(x) dS(x).$$

Due to (8), the operators  $V$  and  $W$  can be also expressed in terms the surface potentials and operators associated with the Laplace operator,

$$(18) \quad V\rho = V_\Delta \left( \frac{\rho}{a} \right),$$

$$(19) \quad W\rho = W_\Delta \rho - V_\Delta \left( \rho \frac{\partial \ln a}{\partial n} \right),$$

We will use relations (18) and (19) to determine the operators  $V$  and  $W$  also for more general spaces for  $\rho$  and, using the corresponding properties for the layer potentials based on a fundamental solution, on Lipschitz domains (cf., e.g., [5]), to obtain, similar to [16, Theorem 3.5], the following mapping and jump properties in Theorems 3 and 4.

**Theorem 3.** *Let  $\Omega$  be a bounded Lipschitz domain. The following operators are continuous if  $\frac{1}{2} < s < \frac{3}{2}$ ,*

$$(20) \quad \mu V : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^s(\mathbb{R}^n), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n);$$

$$(21) \quad r_\Omega W : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^s(\Omega);$$

$$(22) \quad \mu r_{\Omega_-} W : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^s(\Omega_-), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n);$$

$$(23) \quad r_\Omega V : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{1,0}(\Omega; A);$$

$$(24) \quad \mu r_{\Omega_-} V : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{1,0}(\Omega_-; A), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n);$$

$$(25) \quad r_\Omega W : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{1,0}(\Omega; A);$$

$$(26) \quad \mu r_{\Omega_-} W : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{1,0}(\Omega_-; A), \quad \forall \mu \in \mathcal{D}(\mathbb{R}^n);$$

$$(27) \quad \gamma^\pm V : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega);$$

$$(28) \quad \gamma^\pm W : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega);$$

$$(29) \quad T^\pm V : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega);$$

$$(30) \quad T^\pm W : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega).$$

**Theorem 4.** *Let  $\partial\Omega$  be a compact Lipschitz boundary,  $\frac{1}{2} < s < \frac{3}{2}$ ,  $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$  and  $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ . Then*

$$(31) \quad \gamma^+ V\psi - \gamma^- V\psi = 0, \quad \gamma^+ W\varphi - \gamma^- W\varphi = -\varphi;$$

$$(32) \quad T^+ V\psi - T^- V\psi = \psi, \quad T^+ W\varphi - T^- W\varphi = -(\partial_n a)\varphi.$$

Note that the second equation in (32) implies that unlike for the

classical harmonic potential, the conormal derivative of the parametrix-based double layer potential has a jump.

The continuity of operators (27)-(30) in Theorem 3 and the first relation in (31) imply the following assertion.

**Corollary 1.** *Let  $\partial\Omega$  be a compact Lipschitz boundary,  $\frac{1}{2} < s < \frac{3}{2}$ . The following operators are continuous.*

$$(33) \quad \mathcal{V} := \gamma^+ V = \gamma^- V : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega);$$

$$(34) \quad \mathcal{W} := \frac{1}{2}(\gamma^+ W + \gamma^- W) : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^{s-\frac{1}{2}}(\partial\Omega);$$

$$(35) \quad \mathcal{W}' := \frac{1}{2}(T^+ V + T^- V) : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega);$$

$$(36) \quad \mathcal{L} := \frac{1}{2}(T^+ W + T^- W) : H^{s-\frac{1}{2}}(\partial\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega).$$

When the boundary and the density  $\rho$  are smooth enough, the boundary operators defined in Corollary 1 correspond to the boundary integral (pseudodifferential) operators of direct surface values of the single layer potential, the double layer potential  $\mathcal{W}$ , and the co-normal derivatives of the single layer potential  $\mathcal{W}'$  and of the double layer potential, and to the hyper-singular operator, cf. [2, Eq. (3.6)-(3.8)] for the parametrix-based potentials on smooth domains, particularly,

$$\begin{aligned} \mathcal{V}\rho(y) &:= - \int_{\partial\Omega} P(x, y)\rho(x) dS(x), \\ \mathcal{W}\rho(y) &:= - \int_{\partial\Omega} T_x P(x, y)\rho(x) dS(x), \\ \mathcal{W}'\rho(y) &:= - \int_{\partial\Omega} T_y P(x, y)\rho(x) dS(x), \end{aligned}$$

for  $y \in \partial\Omega$ . See also [12, Theorems 7.3, 7.4] about integral representations on Lipschitz domains of the boundary operators associated with the layer potentials, based on fundamental solutions.

Employing definitions (33)-(36), the jump properties (31)-(32) can be re-written as follows.

**Theorem 5.** For  $\psi \in H^{s-\frac{3}{2}}(\partial\Omega)$ , and  $\varphi \in H^{s-\frac{1}{2}}(\partial\Omega)$ ,  $\frac{1}{2} < s < \frac{3}{2}$ ,

$$(37) \quad \gamma^\pm V\psi = \mathcal{V}\psi, \quad \gamma^\pm W\varphi = \mp \frac{1}{2}\varphi + \mathcal{W}\varphi;$$

$$(38) \quad T^\pm V\psi = \pm \frac{1}{2}\psi + \mathcal{W}'\psi, \quad T^\pm W\varphi = \mp \frac{1}{2}(\partial_n a)\varphi + \mathcal{L}\varphi.$$

By Corollary 1 and relations (18)-(19), the operators  $\mathcal{V}, \mathcal{W}, \mathcal{W}'$  and  $\mathcal{L}$  can be expressed in terms their counterparts (provided with the subscript  $\Delta$ ) associated with the Laplace operator,

$$(39) \quad \mathcal{V}\rho = \mathcal{V}_\Delta \left( \frac{\rho}{a} \right),$$

$$(40) \quad \mathcal{W}\rho = \mathcal{W}_\Delta \rho - \mathcal{V}_\Delta \left( \rho \frac{\partial \ln a}{\partial n} \right),$$

$$(41) \quad \mathcal{W}'\rho = a\mathcal{W}'_\Delta \left( \frac{\rho}{a} \right),$$

$$(42) \quad \mathcal{L}\rho = a\mathcal{L}_\Delta \rho - a\mathcal{W}'_\Delta \left( \rho \frac{\partial \ln a}{\partial n} \right).$$

Furthermore, by the Liapunov-Tauber theorem (cf. [5, Lemma 4.1] for the Lipschitz domains),  $\mathcal{L}_\Delta \rho = T_\Delta^+ \mathcal{W}_\Delta \rho = T_\Delta^- \mathcal{W}_\Delta \rho$ .

**Theorem 6.** Let  $S_1$  be a non-empty simply connected subset of the Lipschitz surface  $\partial\Omega$  with a Lipschitz boundary curve and condition (3) holds. Then, the operators

$$(43) \quad \mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega),$$

$$(44) \quad r_{S_1} \mathcal{V} : \tilde{H}^{-\frac{1}{2}}(S_1) \rightarrow H^{\frac{1}{2}}(S_1)$$

are continuously invertible.

*Proof.* We first remark that

$$(45) \quad \langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial\Omega} \geq c \|\psi\|_{H^{-1/2}(\partial\Omega)}, \quad \forall \psi \in H^{-1/2}(\partial\Omega),$$

see e.g. [12, Corollary 8.13]. This evidently gives also

$$(46) \quad \langle \mathcal{V}_\Delta \psi, \psi \rangle_{\partial\Omega} \geq c \|\psi\|_{\tilde{H}^{-1/2}(S_1)}, \quad \forall \psi \in \tilde{H}^{-1/2}(S_1).$$

By the Lax-Milgram lemma, ellipticity estimates (45), (46) and the continuity of operators  $\mathcal{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$  and  $\mathcal{V}_\Delta : \tilde{H}^{-1/2}(S_1) \rightarrow$

$H^{\frac{1}{2}}(S_1)$  imply that these operators are continuously invertible. Relation (18) gives  $\mathcal{V}g = \mathcal{V}_\Delta g^*$ , where  $g^* = g/a$ , which leads to the invertibility of operators (43) and (44).  $\square$

Let us denote

$$(47) \quad \widehat{\mathcal{L}}\rho := a\mathcal{L}_\Delta\rho.$$

Then by (42) and (38), we have,

$$(48) \quad T^\pm W\rho = \mp \frac{1}{2}(\partial_n a)\rho + \widehat{\mathcal{L}}\rho - a\mathcal{W}'_\Delta \left( \rho \frac{\partial \ln a}{\partial n} \right).$$

**Theorem 7.** *Let  $S_1$  be a non-empty simply connected subset of the Lipschitz surface  $\partial\Omega$  with a Lipschitz boundary curve and condition (3) holds. Then, the operator*

$$(49) \quad r_{S_1}\widehat{\mathcal{L}} : \widetilde{H}^{\frac{1}{2}}(S_1) \rightarrow H^{-\frac{1}{2}}(S_1),$$

*is invertible whilst the operators*

$$(50) \quad r_{S_1}(T^\pm W - \widehat{\mathcal{L}}) : \widetilde{H}^{\frac{1}{2}}(S_1) \rightarrow H^{-\frac{1}{2}}(S_1)$$

*are compact.*

*Proof.* Taking into account the invertibility of the operator  $r_{S_1}\mathcal{L}_\Delta : \widetilde{H}^{\frac{1}{2}}(S_1) \rightarrow H^{-\frac{1}{2}}(S_1)$  (see e.g. [29, Eq. (6.39)] together with the Lax-Milgram lemma), (47) implies the invertibility of operator (49).

Now we remark that by (48) and the continuity of operator (35), the operator

$$r_{S_1}(T^\pm W - \widehat{\mathcal{L}}) : \widetilde{H}^{-\frac{1}{2}}(S_1) \rightarrow H^{-\frac{1}{2}}(S_1)$$

is continuous. Then, the Rellich compact embedding theorem implies the compactness of operators (50).  $\square$

**5. Third Green identities and integral relations.** In this section we provide the results similar to the ones in [16] but for our, different, parametrix (8).

Let  $u, v \in H^{1,0}(\Omega; A)$ . Subtracting from the first Green identity (6) its counterpart with the swapped  $u$  and  $v$ , we arrive at the second

Green identity, see e.g. [12],

$$(51) \quad \int_{\Omega} [uAv - vAu] dx = \langle u, T^+v \rangle_{\partial\Omega} - \langle v, T^+u \rangle_{\partial\Omega}.$$

Taking now  $v(x) := P(x, y)$ , and applying (51) to the domain  $\Omega$  without a small vicinity of  $y$ , we obtain by the standard limiting procedures (cf. [24]) the third Green identity for any function  $u \in H^{1,0}(\Omega; A)$ ,

$$(52) \quad u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}Au \quad \text{in } \Omega.$$

If  $u \in H^{1,0}(\Omega; A)$  is a solution of the partial differential equation (5a), then, from (52) we obtain

$$(53) \quad u + \mathcal{R}u - VT^+u + W\gamma^+u = \mathcal{P}f \quad \text{in } \Omega.$$

Taking into account the mapping and jump properties of the potentials from Theorems 2, 3 and 5, we can calculate the traces of the both sides of (53),

$$(54) \quad \frac{1}{2}\gamma^+u + \gamma^+\mathcal{R}u - \mathcal{V}T^+u + \mathcal{W}\gamma^+u = \gamma^+\mathcal{P}f \quad \text{on } \partial\Omega.$$

For some function  $u$  and distributions  $f$ ,  $\Psi$  and  $\Phi$ , we consider a more general, indirect integral relation associated with the third Green identity (53),

$$(55) \quad u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in } \Omega.$$

**Lemma 1.** *Let  $u \in H^1(\Omega)$ ,  $f \in L_2(\Omega)$ ,  $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$  and  $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$  satisfy the relation (55). Then  $u$  belongs to  $H^{1,0}(\Omega, A)$ , it solves the equation*

$$(56) \quad Au = f \quad \text{in } \Omega$$

and the following identity holds true,

$$(57) \quad V(\Psi - T^+u) - W(\Phi - \gamma^+u) = 0 \quad \text{in } \Omega.$$

*Proof.* Since all the potentials in (55) belong to  $H^{1,0}(\Omega; A)$  due to the continuity of operators (15), (16), (23) and (25) in Theorems 2 and 3, equation (55) implies that  $u \in H^{1,0}(\Omega; A)$  as well.

Hence, the third Green identity (52) is valid for the function  $u$ , and we proceed subtracting (52) from (55) to obtain

$$(58) \quad W(\gamma^+u - \Phi) - V(T^+u - \Psi) = \mathcal{P}(Au - f) \text{ in } \Omega.$$

Let us apply the Laplace operator to both sides of equation (58) taking into account relations (10), (18) and (19). Then, we obtain  $Au - f$  in  $\Omega$ , i.e.,  $u$  solves (56). Finally, substituting (56) into (58), we prove (57).  $\square$

**Lemma 2.** *Let  $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ . If*

$$(59) \quad V\Psi^* = 0 \text{ in } \Omega,$$

*then  $\Psi^* = 0$  on  $\partial\Omega$ .*

*Proof.* Taking the trace of (59) gives  $\mathcal{V}\Psi^* = 0$  on  $\partial\Omega$ , which implies the result due to the invertibility of operator (43) in Theorem 6.  $\square$

**6. BDIE system for the mixed problem.** We aim to obtain a segregated boundary-domain integral equation system for mixed BVP (5). To this end, let  $f \in L_2(\Omega)$  and the functions  $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$  and  $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$  be respective continuations of the given boundary data  $\phi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$  and  $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$  to the whole  $\partial\Omega$ , i.e.,  $r_{\partial\Omega_D}\Phi_0 = \phi_0$ ,  $r_{\partial\Omega_N}\Psi_0 = \psi_0$ . Let us now represent

$$(60) \quad \gamma^+u = \Phi_0 + \phi, \quad T^+u = \Psi_0 + \psi, \quad \text{on } \partial\Omega,$$

where  $\phi \in \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  and  $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$  are unknown boundary functions, which we will further consider as formally independent (segregated) of  $u$  in  $\Omega$ .

To obtain one of the possible boundary-domain integral equation systems we employ identity (53) in the domain  $\Omega$ , and identity (54) on  $\partial\Omega$ , substituting there relations (60). Consequently, we obtain the BDIE system (M12) of two equations

$$(61a) \quad u + \mathcal{R}u - V\psi + W\phi = F_0 \quad \text{in } \Omega,$$

$$(61b) \quad \frac{1}{2}\phi + \gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\phi = \gamma^+F_0 - \Phi_0 \quad \text{on } \partial\Omega,$$

for three unknown functions,  $u$ ,  $\psi$  and  $\phi$ . Here

$$(62) \quad F_0 = \mathcal{P}f + V\Psi_0 - W\Phi_0.$$

We remark that  $F_0$  belongs to the space  $H^1(\Omega)$  due to the mapping properties of the surface and volume potentials, see Theorems 2 and 3.

**Theorem 8.** *Let  $f \in L_2(\Omega)$ . Let  $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$  and  $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$  be some fixed extensions of  $\phi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$  and  $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$  respectively.*

(i) *If some  $u \in H^1(\Omega)$  solves the BVP (5), then the triple  $(u, \psi, \phi)^\top \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  where*

$$(63) \quad \phi = \gamma^+ u - \Phi_0, \quad \psi = T^+ u - \Psi_0 \quad \text{on } \partial\Omega,$$

*solves the BDIE system (M12).*

(ii) *If a triple  $(u, \psi, \phi)^\top \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  solves the BDIE system then  $u$  solves the BVP and the functions  $\psi, \phi$  satisfy (63).*

(iii) *System (M12) is uniquely solvable.*

*Proof.* First, let us prove item (i). Let  $u \in H^1(\Omega)$  be a solution of the boundary value problem (5), which implies that  $u \in H^{1,0}(\Omega, A)$ , and let  $\phi, \psi$  be defined by (63). Then, due to (5b) and (5c), we have

$$(\psi, \phi) \in \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N).$$

Then, it immediately follows from the third Green identities (53) and (54) that the triple  $(u, \phi, \psi)$  solves BDIE system  $\mathcal{M}^{12}$ .

Let us prove now item (ii). Let the triple  $(u, \psi, \phi)^\top \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  solve the BDIE system. Taking the trace of equation (61a) and subtracting it from equation (61b), we obtain the first relation in (63). Now, restricting it to  $\partial\Omega_D$ , and taking into account that  $\phi$  vanishes there as  $\text{supp } \phi \subset \overline{\partial\Omega_N}$ , we obtain that  $\phi_0 = \Phi_0 = \gamma^+ u$  on  $\partial\Omega_D$  and, consequently, the Dirichlet condition (5b) of the BVP is satisfied.

We proceed by implementing the Lemma 1 to the first equation, (61a), of system (M12), with  $\Psi = \psi + \Psi_0$  and  $\Phi = \phi + \Phi_0$ . This implies that  $u$  belongs to  $H^{1,0}(\Omega, A)$ , it is a solution of equation (5a) and also

the following equality holds,

$$V(\Psi_0 + \psi - T^+u) - W(\Phi_0 + \phi - \gamma^+u) = 0 \text{ in } \Omega.$$

By virtue of the first relation in (63), the second term of the previous equation vanishes. Hence,

$$V(\Psi_0 + \psi - T^+u) = 0 \quad \text{in } \Omega.$$

Now, by virtue of Lemma 2 we obtain the second relation in (63). Since  $\psi$  vanishes on  $\partial\Omega_N$  and  $\Psi_0 = \psi_0$  on  $\partial\Omega_N$ , the second relation in (63) implies that  $u$  satisfies the Neumann condition (5c).

Item (iii) immediately follows from the uniqueness of the solution of the mixed boundary value problem, cf. Theorem 1, since the zero right-hand side for the corresponding homogeneous BDIE can be considered as given by  $f = 0$ ,  $\Psi_0 = 0$  and  $\Phi_0 = 0$ , cf. (62).  $\square$

BDIE system (61a)-(61b) can be written in the matrix notations as

$$(64) \quad \mathcal{M}^{12}\mathcal{X} = \mathcal{F}^{12},$$

where  $\mathcal{X}$  represents the vector containing the unknowns of the system,

$$\mathcal{X} = (u, \psi, \phi)^\top \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N),$$

the right hand side vector is

$$\mathcal{F}^{12} := [F_0, \gamma^+F_0 - \Phi_0]^\top \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega),$$

and the matrix operator  $\mathcal{M}^{12}$  is

$$(65) \quad \mathcal{M}^{12} = \begin{bmatrix} I + \mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}.$$

**Theorem 9.** *The operator*

$$(66) \quad \mathcal{M}^{12} : H^{1,0}(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^{1,0}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

*is continuous and continuously invertible.*

*Proof.* The continuity of operator (66) is implied by the mapping properties of the operators involved in matrix (65).

To prove invertibility of operator (66), let us consider BDIE system (64) with an arbitrary right hand side  $\tilde{\mathcal{F}} = \{\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2\}^\top \in H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega)$ . From Lemma 6 in the Appendix, we obtain the representation

$$\begin{aligned}\tilde{\mathcal{F}}_1 &= \mathcal{P} f_* + V \Psi_* - W \Phi_* \quad \text{in } \Omega, \\ \tilde{\mathcal{F}}_2 &= \gamma^+ \mathcal{F}_{*1}^{12} - \Phi_* \quad \text{on } \partial\Omega\end{aligned}$$

where the triple

$$(67) \quad (f_*, \Psi_*, \Phi_*)^\top = \tilde{\mathcal{C}}_* \tilde{\mathcal{F}} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is unique and the operator

$$(68) \quad \tilde{\mathcal{C}}_* : H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Applying the equivalence Theorem 8 with

$$f = f_*, \quad \Psi_0 = \Psi_*, \quad \Phi_0 = \Phi_*, \quad \psi_0 = r_{\partial\Omega_N} \Psi_0, \quad \varphi_0 = r_{\partial\Omega_D} \Phi_0,$$

we obtain that the system M12 is uniquely solvable and its solution is

$$u = (A^{DN})^{-1}(f_*, r_{\partial\Omega_D} \Phi_*, r_{\partial\Omega_N} \Psi_*)^\top, \quad \psi = T^+ u - \Psi_*, \quad \phi = \gamma^+ u - \Phi_*$$

while  $r_{\partial\Omega_N} \psi = 0$ ,  $r_{\partial\Omega_D} \phi = 0$ . Here  $(A^{DN})^{-1}$  is the continuous inverse operator to the left-hand-side operator of the mixed BVP (5),  $A^{DN} : H^{1,0}(\Omega; \Delta) \rightarrow L_2(\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega)$ . Representation (67), and continuity of operator (68) complete the proof of invertibility.  $\square$

In the particular case  $a(x) = 1$  at  $x \in \Omega$ , (2) becomes the classical Laplace equation,  $\mathcal{R} = 0$ , and BDIE system (61) splits into the Boundary Integral Equation, BIE,

$$(69) \quad \frac{1}{2}\phi - \mathcal{V}_\Delta \psi + \mathcal{W}_\Delta \phi = \gamma^+ F_{\Delta 0} - \Phi_0 \quad \text{on } \partial\Omega,$$

where  $F_{\Delta 0} = \mathcal{P} f + V_\Delta \Psi_0 - W_\Delta \Phi_0$ , and the representation formula for  $u$  in terms of  $\varphi$  and  $\psi$ ,

$$(70) \quad u = F_0 + V_\Delta \psi - W_\Delta \varphi \quad \text{in } \Omega.$$

Then Theorem 8 leads to the following assertion.

**Corollary 2.** *Let  $a = 1$  in  $\Omega$ ,  $f \in L_2(\Omega)$ , and let  $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$  and  $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$  be some extensions of  $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$  and  $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ , respectively.*

(i) *If some  $u \in H^1(\Omega)$  solves mixed BVP (5) in  $\Omega$ , then the solution is unique, the couple  $(\psi, \varphi) \in \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  given by (63) solves BIE (69), and  $u$  satisfies (70).*

(ii) *If a couple  $(\psi, \varphi) \in \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$  solves BIE (69), then  $u$  given by (70) solves BVP (5) and equations (63) hold. Moreover, BIE (69) is uniquely solvable in  $\tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ .*

BIE (69) can be rewritten in the form

$$(71) \quad \widehat{\mathcal{M}}_{\Delta}^{12} \widehat{\mathcal{U}}_{\Delta} = \widehat{\mathcal{F}}_{\Delta}^{12},$$

where  $\widehat{\mathcal{U}}_{\Delta}^{\top} := (\psi, \varphi) \in \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ ,

$$(72) \quad \widehat{\mathcal{M}}_{\Delta}^{12} := \left[ -\mathcal{V}_{\Delta}, \left( \frac{1}{2} I + \mathcal{W}_{\Delta} \right) \right], \quad \widehat{\mathcal{F}}_{\Delta}^{12} := \gamma^+ F_{\Delta 0} - \Phi_0.$$

**Theorem 10.** *The operator*

$$(73) \quad \widehat{\mathcal{M}}_{\Delta}^{12} : \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$$

*is continuous and continuously invertible.*

*Proof.* The continuity of operator (73) is implied by the mapping properties of the operators involved in the matrix  $\widehat{\mathcal{M}}_{\Delta}^{12}$ .

A solution of BIE (71) with an arbitrary  $\widehat{\mathcal{F}}_{\Delta}^{12} \in H^{\frac{1}{2}}(\partial\Omega)$  is delivered by the couple  $(\psi, \varphi)$  satisfying the extended system

$$(74) \quad \mathcal{M}_{\Delta}^{12} \mathcal{U} = \mathcal{F}_{\Delta 0}^{12},$$

where  $\mathcal{U} = (u, \psi, \varphi)^{\top}$ ,  $\mathcal{F}_{\Delta 0}^{12} = (0, \widehat{\mathcal{F}}_{\Delta}^{12}, )^{\top}$ , and

$$(75) \quad \mathcal{M}_{\Delta}^{12} := \begin{bmatrix} I & -V_{\Delta} & W_{\Delta} \\ 0 & -\mathcal{V}_{\Delta} & \frac{1}{2} I + \mathcal{W}_{\Delta} \end{bmatrix}.$$

The operator  $\mathcal{M}_{\Delta}^{12} : H^{1,0}(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^{1,0}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$  has a continuous inverse due to Theorem 66 for  $a = 1$ . Consequently, operator (73) has a right continuous inverse, which is

also a two-side inverse due to injectivity of operator (73) implied by Corollary 2.  $\square$

Now we prove the counterpart of Theorem 9 in wider spaces.

**Theorem 11.** *The operator*

$$(76) \quad \mathcal{M}^{12} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

*is continuous and continuously invertible.*

*Proof.* The continuity of operator (76) is implied by the mapping properties of the operators involved in matrix (65).

Let now  $\mathcal{M}_0^{12}$  be the matrix operator defined by

$$\mathcal{M}_0^{12} := \begin{bmatrix} I & -V & W_\Delta \\ 0 & -\mathcal{V} & \frac{1}{2}I + \mathcal{W}_\Delta \end{bmatrix}.$$

The operator  $\mathcal{M}_0^{12}$

$$(77) \quad \mathcal{M}_0^{12} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega).$$

is also continuous due to the mapping properties of the operators involved.

Let us prove that operator (77) is invertible. First we remark that due to relation (39) its second line operator can be presented as

$$\mathcal{M}_{02}^{12}(\psi, \varphi)^\top = -\mathcal{V}\psi + \left(\frac{1}{2}I + \mathcal{W}_\Delta\right)\varphi = \mathcal{M}_\Delta^{12} \text{diag}\left(\frac{1}{a}, 1\right)(\psi, \varphi)^\top.$$

Then the continuous invertibility of operator (10) and condition (3) for the coefficient  $a$  imply that the operator

$$\widehat{\mathcal{M}}_{02}^{12} = [-\mathcal{V}, \frac{1}{2}I + \mathcal{W}_\Delta] : \tilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$$

is invertible. Due to the block-triangular structure of operator  $\mathcal{M}_0^{12}$  and obvious invertibility of the identity operator,  $I$ , in  $H^1(\Omega)$ , this, in turn immediately implies invertibility of operator (77).

Further, the operators  $V : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^1(\Omega)$  and  $\mathcal{V} : H^{-\frac{1}{2}}(\partial\Omega) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$  are continuous by Theorem 3 and Corollary 1. Hence, the operators  $V : \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^1(\Omega)$  and  $\mathcal{V} : \tilde{H}^{\frac{1}{2}}(\partial\Omega_N) \rightarrow H^{\frac{1}{2}}(\partial\Omega)$

are compact by the Rellich embedding Theorem. The operator  $\mathcal{R} : H^1(\Omega) \rightarrow H^1(\Omega)$  is also compact by Theorem 2. These compactness properties together with representations (19) and (40) imply that the operator (77) is a compact perturbation of the operator (76), which implies its Fredholm property with index one.

Finally, the Fredholm property and the injectivity of operator  $\mathcal{M}^{12}$ , following from item (iii) of Theorem 8, imply the continuous invertibility of operator (76).  $\square$

## 7. Appendix.

We provide below a simplified version of Lemma 5.5 in [14]. It was proved there for domains with infinitely smooth boundaries but the proof is word-for-word for the Lipschitz domains as well.

**Lemma 3.** *For any function  $\mathcal{F}_0 \in H^{1,0}(\Omega; \Delta)$ , there exists a unique couple  $(f_\Delta, \Psi_\Delta) = \mathcal{C}_0 \mathcal{F}_0 \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$  such that*

$$(78) \quad \mathcal{F}_0 = \mathcal{P}_\Delta f_\Delta + V_\Delta \Psi_\Delta \quad \text{in } \Omega,$$

and  $\mathcal{C}_0 : H^{1,0}(\Omega; \Delta) \rightarrow L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$  is a linear bounded operator.

Employing Lemma 3 for  $\mathcal{F}_0 = \mathcal{F}_1 + W_\Delta \mathcal{F}_2 \in H^{1,0}(\Omega; \Delta)$ , we can easily prove the following assertion (cf. Corollary B.1 in [1].)

**Lemma 4.** *For any couple  $(\mathcal{F}_1, \mathcal{F}_2)^\top \in H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega)$  there exists a unique triple*

$$(f_\Delta, \Psi_\Delta, \Phi_\Delta)^\top = \mathcal{C}_1 (\mathcal{F}_1, \mathcal{F}_2)^\top \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} \mathcal{F}_1 &= \mathcal{P}_\Delta f_\Delta + V_\Delta \Psi_\Delta - W_\Delta \Phi_\Delta \quad \text{in } \Omega, \\ \mathcal{F}_2 &= \Phi_\Delta \quad \text{on } \partial\Omega. \end{aligned}$$

Moreover, the operator

$$\mathcal{C}_1 : H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Employing now Lemma 4 for  $\mathcal{F}_1 = \tilde{\mathcal{F}}_1 \in H^{1,0}(\Omega; \Delta)$  and  $\mathcal{F}_2 = \gamma^+ \tilde{\mathcal{F}}_1 - \tilde{\mathcal{F}}_2 \in H^{\frac{1}{2}}(\partial\Omega)$ , we get the next assertion.

**Lemma 5.** *For any couple  $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^\top \in H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega)$  there exists a unique triple*

$$(f_\Delta, \Psi_\Delta, \Phi_\Delta)^\top = \tilde{\mathcal{C}}_1 (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^\top \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} \tilde{\mathcal{F}}_1 &= \mathcal{P}_\Delta f_\Delta + V_\Delta \Psi_\Delta - W_\Delta \Phi_\Delta \text{ in } \Omega, \\ \tilde{\mathcal{F}}_2 &= \gamma^+ \tilde{\mathcal{F}}_1 - \Phi_\Delta \text{ on } \partial\Omega. \end{aligned}$$

Moreover, the operator

$$\tilde{\mathcal{C}}_1 : H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Finally, denoting  $f_* = af_\Delta$ ,  $\Phi_* = \Phi_\Delta$ ,  $\Psi_* = a\Psi_\Delta - a\Phi_\Delta \frac{\partial \ln a}{\partial n}$ , where  $f_\Delta$ ,  $\Phi_\Delta$  and  $\Psi_\Delta$  are the functions and distributions in Lemma 5, it implies the following statement if we take into account relations (10), (18) and (19).

**Lemma 6.** *For any couple  $(\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^\top \in H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega)$  there exists a unique triple*

$$(f_*, \Psi_*, \Phi_*)^\top = \tilde{\mathcal{C}}_* (\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_2)^\top \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$\begin{aligned} \tilde{\mathcal{F}}_1 &= \mathcal{P} f_* + V \Psi_* - W \Phi_* \text{ in } \Omega, \\ \tilde{\mathcal{F}}_2 &= \gamma^+ \tilde{\mathcal{F}}_1 - \Phi_* \text{ on } \partial\Omega. \end{aligned}$$

Moreover, the operator

$$\tilde{\mathcal{C}}_* : H^{1,0}(\Omega; \Delta) \times H^{\frac{1}{2}}(\partial\Omega) \rightarrow L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

**Conclusions.** A new parametrix for the diffusion equation in a continuously non-homogeneous medium (with variable coefficient) with a Lipschitz boundary has been analysed in this paper. Mapping

properties of the corresponding parametrix-based surface and volume potentials have been shown in corresponding Sobolev spaces.

A BDIE system, based on a new parametrix, for the original BVP has been obtained. The equivalence between the BDIE system and the BVP has been shown along with the invertibility of the matrix operator defining the BDIE system.

Analogous results have been obtained for exterior domains, see [21], following an approach similar to the one in [4].

A generalisation to less smooth coefficients and more general PDE right-hand side can be also consider following [16]. Moreover, these results can be generalised to Bessov spaces as in [3].

Analysing BDIEs for different parametrices, i.e. depending on the variable coefficient  $a(x)$  or  $a(y)$ , is crucial to understand the analysis of BDIEs derived with parametrices that depend on the variable coefficient  $a(x)$  and  $a(y)$  at the same, as it is the case for the Stokes system, see [19, 22, 23].

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